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Generalized Hermitian Algebras

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Abstract We refer to the real Jordan Banach algebra of bounded Hermitian operators on a Hilbert space as a Hermitian algebra. In this paper we define and launch a study of a class of generalized Hermitian (GH) algebras. Among the examples of GH-algebras are ordered special Jordan algebras, JW-algebras, and AJW-algebras, but unlike these more restricted cases, a GH-algebra is not necessarily a Banach space and its lattice of projections is not necessarily complete. In this paper we develop the basic theory of GH-algebras, identify their unit intervals as effect algebras, and observe that their projection lattices are sigma-complete orthomodular lattices. We show that GH-algebras are spectral order-unit spaces and that they admit a substantial spectral theory.

Keywords GH-algebra · Effect · Projection · Orthomodular lattice · Carrier projection · Comparability property · Square root · Absolute value · Spectral resolution · Spectrum

1 Introduction

We shall refer to the real Banach space $\mathbb{G}(\mathfrak{H})$ of bounded Hermitian operators on a Hilbert space $\mathfrak{H}$, organized in the usual way into a partially ordered real vector space, as the *Hermitian algebra* of $\mathfrak{H}$. We call $\mathbb{G}(\mathfrak{H})$ an "algebra" because it is, in fact, a JW-algebra in the sense of [24, p. 3]. Our purpose in this article is to introduce and launch a study of a generalization of $\mathbb{G}(\mathfrak{H})$ which we call a *generalized Hermitian algebra*, or a *GH-algebra* for short.

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Our definition of a GH-algebra (Definition 2.1 below), features a partially ordered abelian group G with order unit 1. In accordance with the theme of [12], we pay special attention to the triple $G \supseteq E \supseteq P$ consisting of the GH-algebra G, the "unit interval" E in G, and the lattice P of "projections" in G. We derive the basic properties of G (Sect. 3), observe that G can be organized into a special Jordan algebra over the real numbers (Theorem 4.2), prove that P is precisely the set of extreme points of the convex set E (Theorem 4.3), and show that P is a σ-complete orthomodular lattice (a σ-OML) [20]. In Sect. 6, we prove that G is a spectral order-unit space [15]; hence, as we observe in Sect. 7, G admits a generalization of the noncommutative spectral theory of Alfsen and Shultz [2, 3].

An ordered special Jordan algebra in the sense of Sarymsakov et al. [23] is a GH-algebra; however, such an algebra is required to satisfy a compatibility condition on the suprema of bounded ascending nets, hence it is a Banach space and its projections form a complete orthomodular lattice. A GH-algebra G is only required to satisfy a much weaker condition on suprema of bounded ascending sequences of pairwise commuting elements (axiom (vii) in Definition 2.1), it is not necessarily a Banach space, and its projection lattice P is not necessarily a complete OML.

The JW-algebras studied by Topping in [24], as well as the AJW-algebras discussed in [24, Sect. 20] are also special cases of GH-algebras. Again, JW-algebras and AJW-algebras are Banach spaces, and in both cases, the projections form complete OMLs.

In [4], a class of Banach-normed Jordan algebras, called JB-algebras, is defined axiomatically and studied by Alfsen et al. For a JB-algebra, a Banach norm is postulated *ab initio*, whereas the (not necessarily Banach) norm on a GH-algebra emerges naturally as a consequence of the theory of archimedean partially ordered real vector spaces [1, Proposition II.1.2].

The theory of ordered Jordan algebras developed by Sarymsakov et al., relies on Freudenthal's spectral theory for semifields. For Topping's JW-algebras and AJW-algebras, the crucial spectral theory comes *gratis* from the spectral theory for operator algebras. The Alfsen-Schultz theory requires a pointwise monotone σ-complete (hence a Banach) order-unit normed space such that exposed faces of the cone base in a dual base-normed space are projective. Our theory of GH-algebras is more algebraic, does not stipulate a Banach norm, does not require a complete projection lattice, is considerably more self-contained than the aforementioned special cases, and yet it too admits a substantial spectral theory.

As per [6], GH-algebras may be regarded as a class of *quantum structures*. Measurable or Borel spaces $(X, \mathcal{B})$ consisting of a nonempty set X and a σ-field $\mathcal{B}$ of subsets of X are featured prominently in the conventional theory of measure and integration, Kolmogorovian probability theory, Mackey's theory of induced unitary representations, and the representation of observables in Dirac-von Neumann quantum mechanics. A σ-field of sets is a σ-complete, but not necessarily a complete, Boolean algebra. The σ-OML of projections in a GH-algebra is covered by maximal Boolean subalgebras, each of which is σ-complete, but not necessarily complete, and each of which corresponds uniquely to a maximal commutative subalgebra of the GH-algebra. In this way, the theory of GH-algebras provides a natural and direct contact with measure, integration, probability, group representation, and the quantum theory of measurement.

2 GH-Algebras

In this article, $\mathbb{N}$ is the set of positive integers and $\mathbb{R}$ is the ordered field of real numbers. We begin with our axioms for a GH-algebra. To help fix ideas, see Examples 2.2, 2.3, and 2.4 below.

Definition 2.1 A *generalized Hermitian* (GH) *algebra* is a subgroup G of the additive group of a ring R with unity 1 such that:

(i) G is a partially ordered abelian group [16, pp. 1–4] with positive cone $G^+ = \{g \in G : 0 \le g\}$ such that $1 \in G^+$ and 1 is an order unit in G [16, p. 4].
(ii) If $a, b \in G^+$, then $ab = ba \Rightarrow ab \in G^+$.
(iii) If $a, b \in G^+$, then $aba \in G^+$ and $aba = 0 \Rightarrow ab = ba = 0$.
(iv) If $g \in G$, then $g^2 \in G^+$.
(v) There exists $d \in G^+$ such that $2d = 1$.
(vi) If $g, h \in G$, then $gh^2g = 0 \Rightarrow gh = hg = 0$.
(vii) Each ascending sequence $g_1 \le g_2 \le \cdots$ of pairwise commuting elements of G that is bounded above in G has a supremum (least upper bound) g in G, and g commutes with every element in G that commutes with every g_i, $i = 1, 2, \ldots$.

The ring R is called the *enveloping ring* of the GH-algebra G, the *unit interval* E in G is defined by $E := \{e \in G : 0 \le e \le 1\}$,[1] and (following Ludwig [22]) elements $e \in E$ are called *effects*. We define $P := \{p \in G : p = p^2\}$ to be the set of all idempotent elements in G, and we refer to the elements $p \in P$ as *projections*.

In the sequel, if j is one of i, ii, ..., or vii, then "axiom (*j*)*" will always refer to axiom* (*j*) *in Definition* 2.1.

Example 2.2 If R is a unital C*-algebra, G is the set of all self-adjoint elements in R, and G is partially ordered as usual [18, 19, Sect. 4.2], then axioms (i)–(vi) are satisfied, but axiom (vii) may fail. If R is a von Neumann algebra, then [18, 19, Lemma 5.1.4] implies that axiom (vii) also holds; hence G is a GH-algebra with enveloping ring R. More generally, if R is an AW*-algebra [21], then G is a GH-algebra with enveloping ring R.

Example 2.3 Let X be a compact Hausdorff space. We denote by $C(X, \mathbb{R})$ the real commutative Banach algebra with pointwise operations and the supremum norm of all continuous functions $f : X \to \mathbb{R}$. With pointwise partial order, and with $R = C(X, \mathbb{R})$ as its enveloping ring, the partially ordered additive abelian group $G = C(X, \mathbb{R})$ satisfies axioms (i)–(vi), but it satisfies axiom (vii) iff[2] X is basically disconnected, i.e., iff the closure of every open F_σ subset of X remains open.[3]

Example 2.4 Let $(X, \mathcal{B})$ be a measurable space. We denote by $R := RV(X, \mathcal{B})$ the real commutative linear algebra with pointwise operations of all random variables (i.e., $\mathcal{B}$-measurable functions) $f : X \to \mathbb{R}$ on $(X, \mathcal{B})$. With pointwise partial order and with R as its enveloping ring, the additive subgroup $RV^b(X, \mathcal{B})$ of R consisting of the bounded random variables on $(X, \mathcal{B})$ is a GH-algebra with R as its enveloping ring.

Let G be a GH-algebra. Arguing as in [11, Lemma 4.1], we see that the element d in axiom (vi) is uniquely determined, and *in what follows, we shall denote it by* $\frac{1}{2} := d$. Thus, $0 \le \frac{1}{2} \in G$, and $\frac{1}{2} \le \frac{1}{2} + \frac{1}{2} = 1$, so $\frac{1}{2} \in E$. The element $\frac{1}{2}$ commutes with every element in R, $g \in G \Rightarrow \frac{1}{2}g \in G$, and the mapping $g \mapsto \frac{1}{2}g$ is a group-theoretic and order automorphism

[1] The notation := means 'equals by definition'.

[2] We use 'iff' as an abbreviation for 'if and only if'.

[3] Recall that a Boolean algebra is σ-complete iff its Stone space is basically disconnected.

of G with $g \mapsto 2g$ as its inverse (cf. [11, Lemma 4.2]). By axioms (iv) and (v), G is closed under the *Jordan product* $\frac{1}{2}(gh+hg)=\frac{1}{2}((g+h)^2-g^2-h^2)$ for all $g,h\in G$.

Standing Assumption 2.5 *In the sequel, we assume that G is a GH-algebra with enveloping ring R,* $G^+=\{a\in G:0\le a\}$ *is the positive cone in G,* $E=\{e\in G:0\le e\le 1\}$ *is the unit interval in G, and* $P=\{p\in G:p=p^2\}$ *is the set of projections in G. To avoid trivialities, we assume that* $G\neq\{0\}$, *i.e.*, $1\neq 0$.

By axiom (i), 1 is an order unit[4] in G, i.e., if $g\in G$, there exists $n\in\mathbb{N}$ such that $g\le n\cdot 1$. Consequently, all of the results in [16] pertaining to partially ordered abelian groups with order units are applicable to G; in particular, G is *directed*, i.e., every element $g\in G$ can be written as $g=a-b$ with $a,b\in G^+$ [16, p. 4].

Evidently,

$$0,1\in P\subseteq E\subseteq G^+\subseteq G\subseteq R.$$

We understand that G^+, E, and P are partially ordered by the restrictions of the partial order $\le$ on G. Since the mappings $g\mapsto -g$, $e\mapsto 1-e$, and $p\mapsto 1-p$ are order-reversing and of period 2 on G, E, and P, respectively, there is a *duality principle* whereby properties of existing suprema in G, E, or P are converted to properties of infima and *vice versa*.

In what follows, we focus attention on the directed partially ordered abelian group G, the unit interval $E\subseteq G$, and the set $P\subseteq E$ of projections. For our purposes in this paper, the enveloping ring R is just a convenient mathematical environment in which to study the triple $P\subseteq E\subseteq G$, and the detailed structure of R will not concern us here.

3 Basic Properties of *G*

Definition 3.1 Let $g,h\in G$. We define gCh to mean that g commutes with h, i.e., that $gh=hg$. If $A\subseteq G$, we also define $C(A)$, called the *commutant of A in G*, by $C(A):=\{g\in G:gCa,\forall a\in A\}$. The subgroup $C(G)$ of G is called the *center* of G. The set $CC(A):=C(C(A))$ is called the *bicommutant of A in G*, and if $g\in CC(h):=CC(\{h\})$, we say that g *double commutes* with h.

In [10, 11, 14] we introduced and studied the notion of an *e-ring*. It is easy to see that (R,E) is an e-ring with $E^+=G^+$; consequently, all of the results in [10, 11, 14] are available to us and we shall use them as necessary in what follows. In the sequel, the properties in the following lemma will be used routinely, often without explicit attribution.

Lemma 3.2 *Let* $g,h,k\in G$. *Then*: (i) $gh=0\Leftrightarrow hg=0$. (ii) $gCh\Rightarrow gh=hg\in G$. (iii) $ghg\in G$. (iv) $n\in\mathbb{N}\Rightarrow g^n\in G$. (v) *If* $0\le k\in C(g)\cap C(h)$, *then* $g\le h\Rightarrow gk\le hk$. (vi) $-1\le g\le 1\Leftrightarrow g^2\le 1$.

Proof To prove (i), suppose $gh=0$. Then $gh^2g=0$, whence by axiom (vi), $hg=0$. Parts (ii)–(iv) follow from [11, Theorem 4.1]. To prove (v), suppose that $0\le k\in C(g)\cap C(h)$ and $g\le h$. Then $0\le h-g,k$ and $(h-g)Ck$, so $0\le(h-g)k=hk-gk$ by axiom (ii), and by part (ii), $hk,gk\in G$.

[4] Some authors use the terminology *strong order unit*.

If $-1 \le g \le 1$, then $0 \le 1-g, 1+g$ with $(1-g)C(1+g)$, whence $0 \le (1-g)(1+g) = 1-g^2$, i.e., $g^2 \le 1$. Conversely, $g^2 \le 1 \Rightarrow -1 \le g \le 1$ follows from [11, Lemma 4.3(iii)], proving (vi). □

The unit interval $E \subseteq G$ forms a so-called *interval effect algebra* [5]. The properties of E and $P \subseteq E$ in the following lemma are consequences of [10, Lemmas 2.4, 2.6, Theorem 2.9, Corollary 2.10] and [11, Theorem 3.2], and they also will be used routinely in the sequel.

Lemma 3.3 *Let $d, e, f \in E$, $p \in P$, and $g, h \in G$. Then*: (i) *If eCf, then $0 \le ef \le e, f \le 1$ and $0 \le e^2 \le e \le 1$.* (ii) *$e \le p \Leftrightarrow e = ep \Leftrightarrow e = pe$ and $p \le e \Leftrightarrow p = pe \Leftrightarrow p = ep$.* (iii) *$pgp, php \in G$, and if $g \le h$, then $pgp \le php$.* (iv) *$d \in P$ iff whenever $e, f, e+f \in E$, then $e, f \le d \Rightarrow e+f \le d$.* (v) *$\{e^n : n \in \mathbb{N}\} \subseteq E$ and $e \ge e^2 \ge e^3 \ge \cdots$.*

Theorem 3.4 *Suppose that $\emptyset \ne Q \subseteq P$ and that Q has a supremum* (*respectively, an infimum*) *p in G. Then $p \in P$ and p is the supremum* (*respectively, the infimum*) *of Q in P.*

Proof By duality it is sufficient to consider the case in which p is the infimum of Q in G. As $0 \le q$ for all $q \in Q$, we have $0 \le p$. Choose any $q_0 \in Q$. Then $0 \le p \le q_0 \le 1$, so $p \in E$. To prove that $p \in P$, we use the criterion in Lemma 3.3(iv). Thus, suppose that $e, f, e+f \in E$ with $e, f \le p$. Then, for all $q \in Q$, we have $e, f \le q$, whereupon $e+f \le q$, and it follows that $e+f \le p$, whence $p \in P$. As $p \in P$, it is clear that p is the infimum of Q in P. □

4 Consequences of Axiom (vii)

As is already evident from Examples 2.2 and 2.3, axiom (vii) is rather strong, but unless G is commutative (i.e., unless $G = C(G)$), it is considerably weaker than the condition that G is monotone σ-complete [16, Chap. 16], [13], which in turn, even in the commutative case, is weaker than the requirement of pointwise monotone σ-completeness [2, (4.1)]. In this section we collect some of the consequences of this critical axiom.

Theorem 4.1 (i) *If $0 \le a \in G$, then 0 is the infimum in G of the sequence $((\frac{1}{2})^n a)_{n\in\mathbb{N}}$.* (ii) *$G$ is archimedean* [16, p. 20]. (iii) *There are no nonzero nilpotents in G.*

Proof (i) As $0 \le a$, the sequence $((\frac{1}{2})^n a)_{n\in\mathbb{N}}$ is descending, bounded below by 0, and its elements commute pairwise, so by axiom (vii) and duality, it has an infimum c in G and $0 \le c$. Also, $c \le (\frac{1}{2})^{n+1}a$ for all $n \in \mathbb{N}$, whence $2c \le (\frac{1}{2})^n a$ for all $n \in \mathbb{N}$, so $2c \le c$, i.e., $c \le 0$, and it follows that $c = 0$.

(ii) Suppose $g, h \in G$ and $ng \le h$ for all $n \in \mathbb{N}$. As G is directed, there exist $a, b \in G$ with $0 \le a, b$ and $h = a - b \le a$, whence $ng \le a$ for all $n \in \mathbb{N}$. In particular, $2^n g \le a$ for all $n \in \mathbb{N}$, and it follows that $g \le (\frac{1}{2})^n a$ for all $n \in \mathbb{N}$. Consequently, by part (i), $g \le 0$.

(iii) As G is archimedean, part (iii) follows from [11, Theorem 4.2]. □

Theorem 4.2 *G can be organized into an archimedean partially ordered real vector space with order unit 1 and it is a* (*special*) *Jordan algebra with respect to the Jordan product $(g, h) \mapsto \frac{1}{2}(gh + hg)$.*

Proof Only axiom (vii) (not the stronger Vigier property used in [11]) is needed for the proof of [11, Theorem 7.2]; hence G can be organized into a partially ordered real vector space that is also a Jordan algebra with the indicated Jordan product. Also G is archimedean by Theorem 4.1(ii). □

By Theorem 4.2, G is not only a special Jordan algebra, it is also a so-called *order-unit space* [1, p. 69], i.e., an archimedean partially ordered real vector space with order unit 1, *and it will be regarded as such in the sequel. As is customary, if* $\lambda \in \mathbb{R}$*, we shall identify* $\lambda \cdot 1$ *with* λ*, so that the ordered field* $\mathbb{R}$ *is identified with a subfield of* $C(G)$.

Theorem 4.3 *The set E of effects in G is convex, and P is precisely the set of extreme points of E.*

Proof The convexity of E is obvious. Suppose $p \in P$ and $p = \lambda e + (1-\lambda) f$ with $0 < \lambda < 1$ and $e, f \in E$. Then $\lambda e, (1-\lambda) f \in E$ with $\lambda e, (1-\lambda) f \leq p$; hence $\lambda e p = \lambda e$ and $(1-\lambda) f p = (1-\lambda) f$, and it follows that $e = ep$ and $f = fp$, i.e., $e, f \leq p$. Thus, $e \leq \lambda e + (1-\lambda) f$, so $(1-\lambda) e \leq (1-\lambda) f$, and therefore $e \leq f$. Likewise, $f \leq e$, so $e = f = p$.

Conversely, suppose $e \in E$ and e is an extreme point of E. As $0 \leq e^2 \leq e \leq 1$, we have $e^2 \in E$. Also, $0 \leq (1-e)^2 = 1 - 2e + e^2$, so $e^2 \leq e \leq 2e \leq 1 + e^2$, whence $0 \leq 2e - e^2 \leq 1$, i.e., $2e - e^2 \in E$. But, $e = \frac{1}{2}e^2 + \frac{1}{2}(2e - e^2)$, and as e is an extreme point of E, it follows that $e = e^2$, i.e., $e \in P$. □

Lemma 4.4 *Let $e \in E$, let $d := 1 - e$, let $d_1 := \frac{1}{2}d$, and define the sequence $(d_n)_{n\in\mathbb{N}}$ recursively by $d_{n+1} := \frac{1}{2}(d + (d_n)^2)$ for all $n \in \mathbb{N}$. Then $(d_n)_{n\in\mathbb{N}}$ is an ascending sequence of pairwise commuting effects in $E \cap CC(e)$, it has a supremum s in G, $s \in CC(\{d_n : n \in \mathbb{N}\}) \subseteq CC(e)$ and $(1-s)^2 = e$ with $1 - s \in CC(e)$.*

Proof By axiom (vii), $(d_n)_{n\in\mathbb{N}}$ has a supremum s in G and $s \in CC(\{d_n : n \in \mathbb{N}\})$. Arguing as in the proof of [11, Theorem 6.1], we have $(1-s)^2 = e$ with $1 - s \in CC(e)$. □

Theorem 4.5 *If $0 \leq g \in G$, there exists a unique element in G, called the square root of g and denoted by $g^{1/2}$, such that $0 \leq g^{1/2}$ and $(g^{1/2})^2 = g$; moreover, $g^{1/2} \in CC(g)$.*

Proof By Theorem 4.1(iii), there are no nonzero nilpotents in G, hence the desired conclusions follow from Lemma 4.4 and [11, Corollary 6.1 and Theorem 6.4]. □

Lemma 4.6 (i) *If $0 \leq g_i \in G$ for $i = 1, 2, \ldots, n$, there exists $0 < \lambda \in \mathbb{R}$ such that $\lambda g_i \in E$ for $i = 1, 2, \ldots, n$.* (ii) $G^+ = \{ne : n \in \mathbb{N}, e \in E\}$.

Proof (i) Because 1 is an order unit in G, there exists $N \in \mathbb{N}$ such that $g_1, g_2, \ldots, g_n \leq N \cdot 1 = N$. Let $\lambda := 1/N$. (ii) If $n \in \mathbb{N}$ and $e \in E$, it is clear that $ne \in G^+$. Conversely, if $g \in G^+$, choose $n \in \mathbb{N}$ with $g \leq n$, and put $e := (1/n)g$. Then $e \in E$ and $g = ne$. □

Theorem 4.7 *Let $g \in G$ be the supremum (respectively, the infimum) in G of the ascending (respectively, descending) sequence $(g_n)_{n\in\mathbb{N}} \subseteq G$ of pairwise commuting elements. Suppose $0 \leq h \in G$ and hCg_n for all $n \in \mathbb{N}$. Then $gh = hg$ is the supremum (respectively, the infimum) in G of $(g_n h)_{n\in\mathbb{N}}$.*

Proof We prove the lemma for an ascending sequence—the result for a descending sequence then follows by duality. By axiom (vii), we have gCh, so $gh = hg \in G$. As $0 \leq g - g_1$ and

$0 \le h$, Lemma 4.6(i) implies that there exists $0 < \lambda \in \mathbb{R}$ such that $\lambda(g - g_1), \lambda h \in E$. For all $n \in \mathbb{N}$, $0 \le g - g_n \le g - g_1$, so $0 \le \lambda(g - g_n) \le \lambda(g - g_1) \le 1$, whence $\lambda(g - g_n), \lambda h \in E$. Also, $\lambda(g - g_n) C \lambda h$, whence $\lambda(g - g_n)\lambda h \le \lambda(g - g_n)$, i.e.,

$$\lambda(g - g_n)h \le g - g_n \quad \text{for all } n \in \mathbb{N}.$$

As $g_n \le g$ and $0 \le h \in C(g_n) \cap C(g)$, it follows that $g_n h \le gh$ for all $n \in \mathbb{N}$. Suppose $k \in G$ and $g_n h \le k$ for all $n \in \mathbb{N}$. We have to show that $gh \le k$. We have

$$\lambda(gh - k) \le \lambda(gh - g_n h) = \lambda(g - g_n)h \le g - g_n \quad \text{for all } n \in \mathbb{N},$$

whence $g_n \le g - \lambda(gh - k)$ for all $n \in \mathbb{N}$, and it follows that $g \le g - \lambda(gh - k)$. Therefore, $\lambda(gh - k) \le 0$, so $gh - k \le 0$, i.e., $gh \le k$. □

Lemma 4.8 *Let $g, h \in G$ with gCh and $0 \le g \le h$. Then*: (i) $g^2 \le h^2$ *and* (ii) $g^{1/2} \le h^{1/2}$.

Proof (i) Follows from [10, Lemma 2.7(iii)].

(ii) Choose $0 < \lambda \in \mathbb{R}$ such that $e := \lambda g \in E$ and $f := \lambda h \in E$. Then eCf, and $e \le f$. As $e^{1/2} = \lambda^{1/2} g^{1/2}$ and $f^{1/2} = \lambda^{1/2} h^{1/2}$, it will be sufficient to prove that $e^{1/2} \le f^{1/2}$. Define

$$d := 1 - e, \qquad c := 1 - f, \qquad d_1 := \frac{1}{2}d, \qquad c_1 := \frac{1}{2}c$$

and by recursion, for all $n \in \mathbb{N}$,

$$d_{n+1} := \frac{1}{2}(d + (d_n)^2) \quad \text{and} \quad c_{n+1} := \frac{1}{2}(c + (c_n)^2).$$

By Lemma 4.4, $(d_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ have suprema s and t, respectively, in G; moreover, $e^{1/2} = 1 - s$ and $f^{1/2} = 1 - t$. As $e \le f$, we have $c \le d$, $c_1 \le d_1$, and by part (i) and induction on n, $c_n \le d_n$ for all $n \in \mathbb{N}$. Therefore, $t \le s$, so $e^{1/2} = 1 - s \le 1 - t = f^{1/2}$. □

5 Carrier Projections and Positive Parts

The notion of a *carrier projection* (Definition 5.3 below) enables us to deal efficiently with the question of whether two elements $g, h \in G$ annihilate each other. By Lemma 3.2(i), $gh = 0 \Leftrightarrow hg = 0$, so it is not necessary to distinguish between left and right annihilation.

Lemma 5.1 *Let $e \in E$. Then $((1 - e)^n)_{n\in\mathbb{N}}$ is a descending sequence of pairwise commuting effects in E, whence by axiom* (vii), *it has an infimum q in G and $q \in CC(e)$. Moreover, $1 - q \in P \cap CC(e)$, and for all $h \in G$, $eh = 0 \Leftrightarrow (1 - q)h = 0$.*

Proof By axiom (vii), $q \in CC\{(1 - e)^n : n \in \mathbb{N}\} \subseteq CC(e)$, so $1 - q \in CC(e)$. Evidently, $0 \le q \le 1 - e \le 1$, whence $0 \le q^{1/2} \le 1$ by Lemma 4.8(ii), i.e., $q^{1/2} \in E$, and it follows that $q = (q^{1/2})^2 \le q^{1/2}$. For every $n \in \mathbb{N}$, we have $q \le (1 - e)^{2n}$, so by Lemma 4.8(ii) again, $q^{1/2} \le (1 - e)^n$, and therefore $q^{1/2} \le q$. Consequently, $q^{1/2} = q$, so $q = q^2 \in P$, whence $1 - q \in P \cap CC(e)$.

Suppose $h \in G$ and $eh = 0$. Then $0 \le h^2$ and hCe, therefore $h^2 C(1 - e)^n$ for all $n \in \mathbb{N}$. By Theorem 4.7, $h^2 q = qh^2$ is the infimum in G of the sequence $(h^2(1 - e)^n)_{n\in\mathbb{N}}$. But $h^2(1 - e) = h^2$, and by induction on n, $h^2(1 - e)^n = h^2$ for all $n \in \mathbb{N}$, so all terms in the sequence $(h^2(1-e)^n)_{n\in\mathbb{N}}$ are equal to h^2, and it follows that $h^2q=h^2$. Therefore, $(1-q)h^2(1-q)=0$, so $(1-q)h=0$ by axiom (vi), and it follows that $eh=0 \Rightarrow (1-q)h=0$.

Conversely, suppose that $(1-q)h=0$. As q is the infimum in G of $((1-e)^n)_{n\in\mathbb{N}}$, we have $q\le 1-e$, so $e\le 1-q\in P$, and it follows that $e=e(1-q)$. Therefore $eh=e(1-q)h=0$, and we have $eh=0\Leftrightarrow (1-q)h=0$. □

Theorem 5.2 *For each $g\in G$ there is a uniquely determined projection $g^{\mathrm{o}}\in P$ such that, for all $h\in G$, $gh=0\Leftrightarrow g^{\mathrm{o}}h=0$. Moreover, $g^{\mathrm{o}}\in CC(g)$.*

Proof Let $g\in G$. As $0\le g^2$, there exists $0<\lambda\in\mathbb{R}$ such that $e:=\lambda g^2\in E$. By Lemma 5.1, there is a projection $g^{\mathrm{o}}\in P\cap CC(e)=CC(g^2)\subseteq CC(g)$ such that, for all $h\in G$, $eh=0\Leftrightarrow g^{\mathrm{o}}h=0$. For all $h\in G$, axiom (vi) implies that $gh=0\Rightarrow g^2h=0\Rightarrow hg^2h=0\Rightarrow gh=0$, so

$$gh=0 \quad\Leftrightarrow\quad g^2h=0 \quad\Leftrightarrow\quad \lambda g^2h=0 \quad\Leftrightarrow\quad eh=0 \quad\Leftrightarrow\quad g^{\mathrm{o}}h=0.$$

To prove uniqueness, suppose $p\in P$ and $gh=0\Leftrightarrow ph=0$ for all $h\in G$. Then $g^{\mathrm{o}}h=0\Leftrightarrow ph=0$ for all $h\in G$. Putting $h=1-p$, we find that $g^{\mathrm{o}}(1-p)=0$, i.e., $g^{\mathrm{o}}=g^{\mathrm{o}}p$, so $g^{\mathrm{o}}\le p$. By symmetry, $p\le g^{\mathrm{o}}$, so $p=g^{\mathrm{o}}$. □

Definition 5.3 If $g\in G$, the uniquely determined projection g^{o} in Theorem 5.2 is called the *carrier projection* of g.

As left and right annihilation are equivalent in G, the carrier projection $g^{\mathrm{o}}\in P$ of $g\in G$ is characterized not only by the condition $gh=0\Leftrightarrow g^{\mathrm{o}}h=0$ for all $h\in G$, but also by the condition $hg=0\Leftrightarrow hg^{\mathrm{o}}=0$ for all $h\in G$. Therefore, G has the so-called *carrier property* [14, Definition 3.3], and the results of [14, Sect. 3] are at our disposal.

Theorem 5.4 *P is a σ-complete orthomodular lattice* (σ*-OML*).

Proof That P is an OML follows from [14, Theorem 3.5]. Let $p_1\le p_2\le\cdots$ be an ascending sequence in P. To prove that P is σ-complete, it will be sufficient to show that $(p_n)_{n\in\mathbb{N}}$ has a supremum in P. As comparable projections commute, the projections in the sequence $(p_n)_{n\in\mathbb{N}}$ commute pairwise, whence by axiom (vii), $(p_n)_{n\in\mathbb{N}}$ has a supremum p in G. By Theorem 3.4, $p\in P$ and p is the supremum of $(p_n)_{n\in\mathbb{N}}$ in P. □

If $p,q\in P$, we use the usual notation $p\vee q$ and $p\wedge q$ for the supremum and infimum, respectively, of p and q in the σ-OML P. More generally, if $Q\subseteq P$, we use the notation $\bigvee Q$ (respectively, $\bigwedge Q$) for an existing supremum (respectively, infimum) of Q in P. By definition, p is *orthogonal to* q, in symbols $p\perp q$ iff $p\le 1-q$, i.e., iff $p+q\in P$. Recall that elements $p,q\in P$ are called *Mackey compatible* iff there are pairwise orthogonal elements $p_1,q_1,r\in P$ such that $p=p_1\vee r$ and $q=q_1\vee r$.

Lemma 5.5 *Let $p,q\in P$. Then*: (i) $p\le q\Leftrightarrow q-p\in P$. (ii) *If $p\le q$, then $q-p=q\wedge(1-p)$*. (iii) $p\perp q\Leftrightarrow pq=qp=0$ *and if $p\perp q$, then $p+q=p\vee q$*. (iv) *If pCq, then $p\vee q=p+q-pq$, $p\wedge q=pq$, and $p+q=p\vee q+p\wedge q$*. (v) *p and q are Mackey compatible iff pCq*.

Proof For (i) and (ii), see [10, Theorem 2.9 and Corollary 2.14]. For (iii), see [10, Theorem 2.11 and Corollary 2.13]. For (iv), see [10, Theorem 2.12 and Corollary 2.13]. To prove (v), first suppose pCq. By (iv), $r := pq = p \wedge q \in P$, by (ii), $p - r, q - r \in P$; hence $p = (p - r) + r = (p - r) \vee r$, $q = (q - r) + r = (q - r) \vee r$, and $p - r, q - r, r$ are pairwise orthogonal. Conversely, if $p = p_1 + r$, $q = q_1 + r$ and p_1, q_1, r are pairwise orthogonal elements in P, then $pq = qp = r$. □

Definition 5.6 Let $g \in G$. As $0 \le g^2$ (axiom (iv)), we can and do define $|g| := (g^2)^{1/2}$. Also, we define $g^+ = \frac{1}{2}(|g| + g)$ and $g^- = \frac{1}{2}(|g| - g)$.

Using the carrier projection $p = (g^+)^{\mathrm{o}}$, we now show that the *absolute value* $|g|$ and the *positive* and *negative* parts g^+ and $-g^-$ of $g \in G$ have the expected properties.

Theorem 5.7 *Let $g \in G$ and let $p := (g^+)^{\mathrm{o}}$. Then*:

(i) $\lvert g\rvert^2 = g^2$.	(ii) $\lvert g\rvert, g^+, g^- \in CC(g)$.
(iii) $g = g^+ - g^-$.	(iv) $0 \le \lvert g\rvert = g^+ + g^-$.
(v) $g^+g^- = g^-g^+ = 0$.	(vi) $\lvert -g\rvert = \lvert g\rvert$.
(vii) $g^- = (-g)^+$.	(viii) $g^+ = (-g)^-$.
(ix) $p \in CC(g)$	(x) $pC\lvert g\rvert$
(xi) $pg = g^+$.	(xii) $(1 - p)g = -g^-$.
(xiii) $0 \le p\lvert g\rvert = g^+$.	(xiv) $0 \le (1 - p)\lvert g\rvert = g^-$.

Proof (i)–(viii) are obvious. By Theorem 5.2 and (ii), we have $p \in CC(g^+) \subseteq CC(g)$, proving (ix), and (x) follows from (ix) and (ii). We have $pg^+ = g^+$, and since $g^+g^- = 0$, we also have $pg^- = 0$; hence (xi) and (xii) follow from $g = g^+ - g^-$. Likewise, $p|g| = g^+$ and $(1 - p)|g| = g^-$ follow from $|g| = g^+ + g^-$. Since $0 \le |g|, p, 1 - p$, axiom (ii) implies that $0 \le p|g| = g^+$ and $0 \le (1 - p)|g| = g^-$, proving (xiii) and (xiv). □

Corollary 5.8 *If $g \in G$, then g^+ and g^- are characterized by the properties $g = g^+ - g^-$, $g^+g^- = 0$, and $0 \le g^+ + g^-$.*

Proof Suppose $a, b \in G$, $g = a - b$, $ab = 0$, and $0 \le a + b$. Then $ab = ba = 0$, whence $g^2 = a^2 + b^2 = (a + b)^2$, and as $0 \le a + b$, it follows that $a + b = (g^2)^{1/2} = |g|$. Therefore, $g^+ = \frac{1}{2}(|g| + g) = \frac{1}{2}(a + b + a - b) = a$ and $g^- = \frac{1}{2}(|g| - g) = \frac{1}{2}(a + b - a + b) = b$. □

6 Compressions, the Projection Cover Property, and the Comparability Property

A mapping $J : G \to G$ is called a *retraction* with *focus* p iff, for all $g, h \in G$ and all $e \in E$, (i) $J(1) = p \in E$, (ii) $J(g + h) = J(g) + J(h)$, (iii) $g \le h \Rightarrow J(g) \le J(h)$, and (iv) $e \le p \Rightarrow J(e) = e$ [7, Definition 2.1]. A retraction J with focus p is called a *compression* iff, for all $e \in E$, $J(e) = 0 \Rightarrow e \le 1 - p$ [7, Definition 2.4].

Clearly, each projection $p \in P$ determines a retraction J_p on G with focus p according to $J_p(g) := pgp$ for all $g \in G$. Conversely, by [7, Theorem 4.5], each retraction J on G is a compression and has the form $J = J_p$ where $p = J(1) \in P$. Thus, *there is a bijective correspondence $p \leftrightarrow J_p$ between projections $p \in P$ and compressions J_p on G*. As a consequence of [9, Theorem 1], the family $(J_p)_{p \in P}$ constitutes a so-called *compression base* for G [9, Definition 2].

If $p \in P$ and $g \in G$, it is easily seen that $gp = pg$ iff $g = J_p(g) + J_{1-p}(g)$, hence the notation $g \in C(p)$ in [9, Definition 3] and [15, Definition 1.5] agrees with the notation in

Definition 3.1. As per [15, Definition 1.5(iii)], $CPC(g)$ denotes the set of all elements $h \in G$ such that, for all $p \in P$, $g \in C(p) \Rightarrow h \in C(p)$, i.e., all elements $h \in G$ that commute with every projection p that commutes with g. Clearly, $CC(g) \subseteq CPC(g)$.

If $g \in G$, then, translating [15, Definition 1.6(i)] into our present context, we have

$$P^{\pm}(g) := \{p \in P \cap CPC(g) : gp = pg \text{ and } (1-p)g \leq 0 \leq pg\}.$$

Remark 6.1 By parts (ix) and (xi)–(xiv) of Theorem 5.7, $(g^+)^{\mathrm{o}} \in P^{\pm}(g)$, so G has the *comparability property* [15, Definition 1.6(ii)].[5] By [14, Lemma 3.4(iv)], if $e \in E$, then e^{o} is the smallest projection $p \in P$ such that $e \leq p$; consequently, G has the so-called *projection cover property* [15, Definition 1.4]. Since the order-unit space G has both the projection cover and comparability properties, it is a so-called *spectral order-unit space* [15, Definition 1.7], and if we rewrite $J_p(g)$ as pgp (or as pg if $g \in C(p)$), all of the results in [15] pertaining to spectral order-unit spaces are applicable to G.

Remark 6.2 By [15, Theorem 2.1], there is a uniquely determined mapping $' : G \to P$, called the *Rickart mapping*, such that, for all $g \in G$ and all $p \in P$, $p \leq g' \Leftrightarrow pg = gp = 0$. Obviously, $g' = 1 - g^{\mathrm{o}}$ and $g^{\mathrm{o}} = g'' := (g')'$, so in our present context, the Rickart mapping has the even stronger property that, for all $h \in G$, $gh = 0 \Leftrightarrow hg = 0 \Leftrightarrow h = g'h \Leftrightarrow h = hg'$. Thus, if we rewrite g' as $1 - g^{\mathrm{o}}$ and g'' as g^{o}, all of the results in [15] pertaining to the Rickart mapping are applicable to G.

According to [15, Definition 2.1], the positive part of $g \in G$ is gp for any choice of $p \in P^{\pm}(g)$; hence by Theorem 5.7(xi), our notation g^+, g^-, $|g|$ as per Definition 5.6 agrees with the corresponding notation in [15].

Theorem 6.3 *Let $g \in G$ and let $s := (g^+)^{\mathrm{o}} - (g^-)^{\mathrm{o}}$. Then* (i) $s \in CC(g)$. (ii) $g^{\mathrm{o}} = s^2$. (iii) $|g| = sg = gs$. (iv) $g = s|g| = |g|s$. (v) $|g|^{\mathrm{o}} = g^{\mathrm{o}}$.

Proof By Theorem 5.7(ix), $(g^+)^{\mathrm{o}} \in CC(g)$. Likewise, by Theorem 5.7 $(g^-)^{\mathrm{o}} = ((-g)^+)^{\mathrm{o}} \in CC(-g) = CC(g)$, and (i) follows. See [14, Lemma 4.4 and Theorem 4.7(iii)] for proofs of (ii), (iii), and (iv). To prove (v), we note that $gh = 0 \Rightarrow sgh = 0 \Rightarrow |g|h = 0 \Rightarrow s|g|h = 0 \Rightarrow gh = 0$, so $gh = 0 \Leftrightarrow |g|h = 0$. □

The element s in Theorem 6.3 is called the *signum* of g, and the equation $g = s|g| = g|s|$ is called the *polar decomposition* of g.

By [1, Proposition II.1.2], the order-unit space G is a normed real vector space with the *order-unit norm*

$$\|g\| = \inf\{\lambda \in \mathbb{R} : 0 < \lambda \text{ and } -\lambda \leq g \leq \lambda\} \quad \text{for every } g \in G.$$

Theorem 6.4 *Let $g, h \in G$ and $p \in P$. Then*: (i) $-\|g\| \leq g \leq \|g\|$. (ii) $-h \leq g \leq h \Rightarrow \|g\| \leq \|h\|$. (iii) $\|g^2\| = \|g\|^2$. (iv) $0 \neq p \Rightarrow \|p\| = 1$. (v) $\|pgp\|\| \leq \|g\|$. (vi) $h = |g| \Rightarrow \|h\| = \|g\|$. (vii) $0 \leq g, h \Rightarrow \|g - h\| \leq \max\{\|g\|, \|h\|\}$. (viii) $\|\frac{1}{2}(gh + hg)\| \leq \|g\|\|h\|$. (ix) $gCh \Rightarrow \|gh\| \leq \|g\|\|h\|$.

[5] In [8, Definition 3.4] the comparability property was called *general comparability* because, for interpolation groups, it is equivalent to the property of the same name [16, Chap. 8].

Proof Part (i) follows from [1, Proposition II.1.2], and part (ii) follows from [16, Proposition 7.12(c)]. If $0 < \lambda \in \mathbb{R}$, then Lemma 3.2(vi) with g replaced by $\lambda^{-1}g$ implies that $-\lambda \le g \le \lambda \Leftrightarrow g^2 \le \lambda^2$, from which (iii) follows. As $p^2 = p$, (iv) follows from (iii). Part (i) implies that $-\|g\|p = -p\|g\|p \le pgp \le p\|g\|p = \|g\|p$, and since $p \le 1$, $\|g\|p \le \|g\|$, whence $-\|g\| \le pgp \le \|g\|$ and therefore (v) holds. If $h = |g|$, then $h^2 = g^2$, so $\|h\|^2 = \|g\|^2$ by (iii), proving (vi).

To prove (vii), we assume without loss of generality that $0 < \|h\| \le \|g\|$. As $0 \le g$, (i) implies that $h \le \|h\| \le \|g\| \le \|g\| + g$. Likewise, $g \le \|g\| \le \|g\| + h$, and it follows that $-\|g\| \le g - h \le \|g\|$, whence $\|g - h\| \le \|g\| = \max\{\|g\|, \|h\|\}$.

To prove (viii), we begin by assuming without loss of generality that $g, h \ne 0$. Let $a := g/\|g\|$ and $b := h/\|h\|$, so that $\|a\| = \|b\| = 1$ and $\|\frac{1}{2}(gh + hg)\| = \|g\|\|h\|\|\frac{1}{2}(ab + ba)\|$. Thus, it will be sufficient to prove that $\|\frac{1}{2}(ab + ba)\| \le 1$. As $\|a\| = \|b\| = 1$, we have $\|a \pm b\| \le 2$, so by (iii),

$$\|(a \pm b)^2\| = \|a \pm b\|^2 \le 4.$$

Therefore, by (vii),

$$\left\|\frac{1}{2}(ab = ba)\right\| = \frac{1}{4}\|(a+b)^2 - (a-b)^2\| \le \frac{1}{4}\max\{\|(a+b)^2\|, \|(a-b)^2\|\} \le 1.$$

Obviously, (ix) follows from (viii). □

Recall that G is said to be *monotone σ-complete* iff every ascending sequence in G that is bounded above in G has a supremum in G [13], [16, Chap. 16]. See [17, Proposition 3.9] for a proof of the following.

Theorem 6.5 *If G is monotone σ-complete, then it is a real Banach space under the order-unit norm.*

Lemma 6.6 *Suppose that $a \in G$, $g_1 \le g_2 \le \cdots$ is an ascending sequence in G, and $g_n \to a$ in norm. Then*: (i) *a is an upper bound in G for $(g_n)_{n\in\mathbb{N}}$.* (ii) *If $(g_n)_{n\in\mathbb{N}}$ has a supremum b in G, then $a = b$.* (iii) *If the elements of $(g_n)_{n\in\mathbb{N}}$ commute pairwise, then a is the supremum of $(g_n)_{n\in\mathbb{N}}$ and $a \in CC(\{g_n : n \in \mathbb{N}\}$.*

Proof For each $m \in \mathbb{N}$, choose $n_m \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$, $n_m \le n \Rightarrow \|g_n - a\| \le 1/m$. Then, by Theorem 6.4(i), $n_m \le n \in \mathbb{N} \Rightarrow g_n - a \le 1/m \Rightarrow g_n \le a + 1/m$. As $g_1 \le g_2 \le \cdots \le g_{m_n}$, it follows that $g_n \le a + 1/m$, whence $m(g_n - a) \le 1$ for all $m, n \in \mathbb{N}$. Since G is archimedean, we deduce that $g_n - a \le 0$, i.e., $g_n \le a$ for all $n \in \mathbb{N}$, proving (i). Now assume that $(g_n)_{n\in\mathbb{N}}$ has a supremum b in G. By (i), we have $g_n \le b \le a$ for all $n \in \mathbb{N}$, whence $0 \le b - g_n \le a - g_n$, and it follows from Theorem 6.4(ii) that $\|b - g_n\| \le \|a - g_n\|$. Therefore, $g_n \to b$ in norm, and consequently, $a = b$, proving (ii). Part (iii) follows from (i), (ii), and axiom (vii). □

7 Spectral Resolution and the Spectrum

As G is a spectral order-unit space (Remark 6.1), all of the results in [15, Sects. 3 and 4] pertaining to the spectral resolution and the spectrum are available for G. We begin by adapting the notation in [15], as per Remarks 6.1 and 6.2, to our present context by rewriting $J_p(g)$ as pgp (or as pg if pCg), g' as $1-g^{\mathrm{o}}$, and g'' as g^{o} for $g \in G$ and $p \in P$.

If $g \in G$ and $\lambda \in \mathbb{R}$, we define the projections

$$p_\lambda := 1 - ((g-\lambda)^+)^{\mathrm{o}} \quad \text{and} \quad d_\lambda := 1-(g-\lambda)^{\mathrm{o}}.$$

The family $(p_\lambda)_{\lambda\in\mathbb{R}}$ is called the *spectral resolution of* g, and the projection d_λ is called the λ*-eigenprojection for* g [15, Definition 3.2]. If $\lambda \in \mathbb{R}$, then λ is an *eigenvalue* of g iff $d_\lambda \neq 0$. We also define the *lower* and *upper spectral bounds for* g by

$$L := \sup\{\lambda \in \mathbb{R} : \lambda \le g\} \quad \text{and} \quad U := \inf\{\lambda \in \mathbb{R} : g \le \lambda\},$$

respectively [15, Definition 3.1]. By [15, Theorem 3.1], $-\infty < L \le U < \infty$ and $\|g\| = \max\{|L|, |U|\}$.

Standing Assumption 7.1 *In what follows, we assume that* $(p_\lambda)_{\lambda\in\mathbb{R}}$ *is the spectral resolution of* g, $(d_\lambda)_{\lambda\in\mathbb{R}}$ *is the family of eigenprojections for* g, *and the lower and upper spectral bounds for* g *are* L *and* U, *respectively.*

Theorem 7.2 *Let* $\lambda, \mu \in \mathbb{R}$. *Then*:

(i) $p_\lambda, d_\lambda \in P \cap CC(g)$[6] *and* $d_\lambda C p_\lambda$.
(ii) $p_\lambda g - \lambda p_\lambda \le 0 \le (1-p_\lambda)g - \lambda(1-p_\lambda)$.
(iii) $\lambda \le \mu \Rightarrow p_\lambda \le p_\mu$ *and* $p_\mu - p_\lambda = p_\mu \wedge (1-p_\lambda)$.
(iv) $\lambda < \mu \Rightarrow d_\lambda \le p_\lambda \le 1 - d_\mu \Rightarrow d_\lambda d_\mu = 0$.
(v) $\mu \ge U \Rightarrow p_\mu = 1$, *and* $\lambda < U \Rightarrow p_\lambda < 1$.
(vi) $\lambda < L \Rightarrow p_\lambda = 0$, *and* $L < \mu \Rightarrow 0 < p_\mu$.
(vii) $L = \sup\{\lambda \in \mathbb{R} : p_\lambda = 0\}$, *and* $U = \inf\{\mu \in \mathbb{R} : p_\mu = 1\}$.
(viii) *If* $\lambda \le \mu$ *and* $q \in P$ *with* $q \le p_\mu - p_\lambda$, *then* $\lambda q \le qgq \le \mu q$.
(ix) $p_\lambda = \bigwedge\{p_\mu : \lambda < \mu \in \mathbb{R}\}$.
(x) $p_\mu - d_\mu = \bigvee\{p_\lambda : \mu > \lambda \in \mathbb{R}\}$.

Proof Part (i) follows from Theorems 5.2 and 5.7(ii). Parts (ii)–(iv) and (vi)–(viii) follow from the corresponding parts of [15, Theorem 3.3], and part (v) follows from [15, Theorem 3.3(v)] together with the fact that, by part (ix), $p_U = 1$. Parts (ix) and (x) are consequences of [15, Theorems 3.5, 3.6]. □

Applying the fact that $p_U = 1$ to [15, Theorem 3.4], obtain the following fundamental result:

Theorem 7.3 *The element* $g \in G$ *can be written as a norm-convergent Riemann-Stieltjes type integral*

$$g = \int_{L-0}^{U} \lambda \, dp_\lambda.$$

Theorem 7.4 *There exists an ascending sequence* $g_1 \le g_2 \le \cdots$ *in* $CC(g)$ *such that each* g_n *is a finite linear combination of projections in the family* $(p_\lambda)_{\lambda\in\mathbb{R}}$ *and* $g_n \to g$ *in norm. Moreover,* g *is the supremum of* $(g_n)_{n\in\mathbb{N}}$ *in* G *and* $g \in CC(\{g_1, g_2, \ldots\})$.

[6] In [15], only the weaker conditions $p_\lambda, d_\lambda \in P \cap CPC(g)$ were available.

Proof The first part of the theorem follows from [15, Corollary 3.1] together with the observation that each p_λ belongs not only to $CPC(g)$, but to $CC(g)$. The second part then follows from Lemma 6.6. □

Lemma 7.5 *If $h \in G$, then $hCg \Leftrightarrow hCp_\lambda$ for all $\lambda \in \mathbb{R}$.*

Proof If hCg and $\lambda \in \mathbb{R}$, then since $p_\lambda \in CC(g)$, it follows that hCp_λ. Conversely, if hCp_λ for all $\lambda \in \mathbb{R}$, then for the ascending sequence $(g_n)_{n\in\mathbb{N}}$ in Theorem 7.4, we have $h \in C(\{g_1, g_2, \ldots\})$, whence hCg. □

Theorem 7.6 *Let $g, h \in G$ and let $A \subseteq G$. Then*: (i) *gCh iff every projection in the spectral resolution of g commutes with every projection in the spectral resolution of h*. (ii) $C(C(A) \cap P) = CC(A)$. (iii) $CPC(g) = CC(g)$.

Proof (i) Follows from Lemma 7.5. As $C(A) \cap P \subseteq C(A)$, we have $CC(A) \subseteq C(C(A) \cap P)$. Conversely, suppose $g \in C(C(A) \cap P)$, $h \in C(A)$, and $(q_\lambda)_{\lambda\in\mathbb{R}}$ is the spectral resolution of h. Then by Lemma 7.5, $q_\lambda \in C(A) \cap P$, so gCq_λ for every $\lambda \in \mathbb{R}$, and therefore gCh. Consequently, $C(C(A) \cap P) \subseteq CC(A)$, and (ii) holds. Putting $A := \{g\}$ in (ii), we obtain (iii). □

Definition 7.7 Let $A \in G$, $\rho \in \mathbb{R}$. We say that ρ belongs to the *resolvent set* of a iff there exists $0 < \epsilon \in \mathbb{R}$ such that p_λ is constant for λ in the open interval $(\rho - \epsilon, \rho + \epsilon)$. The *spectrum* of a, in symbols, spec(a), is defined to be the complement in $\mathbb{R}$ of the resolvent set of a.

The following result is a consequence of [15, Theorem 4.2].

Theorem 7.8 *Every isolated point of* spec(a) *is an eigenvalue of a and every eigenvalue of a belongs to* spec(a).

Theorem 7.9 (i) *If $\gamma, \mu \in \mathbb{R}$, then* $\text{spec}(\gamma g + \mu) = \{\gamma\alpha + \mu : \alpha \in \text{spec}(g)\}$. (ii) spec$(g)$ *is a closed nonempty subset of the closed interval* $[L, U] \subseteq \mathbb{R}$. (iii) $L = \inf(\text{spec}(g)) \in \text{spec}(g)$, $U = \sup(\text{spec}(g)) \in \text{spec}(g)$, *and* $\|g\| = \sup\{|\alpha| : \alpha \in \text{spec}(g)\}$. (iv) $0 \le g \Leftrightarrow \text{spec}(g) \subseteq [0, \infty)$.

Proof By [15, Theorem 4.1], we have $\text{spec}(a + \mu) = \{\alpha + \mu : \alpha \in \text{spec}(a)\}$ and $\text{spec}(-a) = \{-\alpha : \alpha \in \text{spec}(a)\}$; hence (i) follows from the obvious facts that, for $0 \le \gamma$, $(\gamma a)^+ = \gamma a^+$ and for $0 \ne \gamma$, $(\gamma a)^{\text{o}} = a^{\text{o}}$. Parts (ii), (iii), and (iv) follow from [15, Theorems 4.3 and 4.4]. □

Remark 7.10 Although we shall not do so here, it can be shown that spec(g) is the set of all $\lambda \in \mathbb{R}$ such that $g - \lambda$ fails to have a multiplicative inverse in G (cf. [15, Theorem 4.7]).

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